Probability, Part 1

The Three Stylized Experiments

To make the following ideas concrete, we'll consider The Three Stylized Prcesses below:

1. **The Unbiased Coin**: Toss The Unbiased Coin. What is the probability of obtaining a head?
2. **The Three-Card Deck**: Draw two cards from The Three-Card Deck, which contains the 3♣, the 9♢, and the A♢.
    a. **base probabilities**: What is the probability that the first card is a 3? 9? A? ♣? ♢? What about the second card?
    b. **conditional probabilities** What is the probability that the second card is an A, given that the first card is a ♢? Given that the first card is a ♣? What is the probability that the second card is a three, given that the first card is a ♢? There are many more combinations here.
    c. **AND probabilities** What is the probability that the first card is a 3 and the second card is a 9? What is the probability that the first card is a 3 and the second card is a ♣?
3. **The Urn**: Draw k marbles at random from The Urn, which contains r red marbles, b blue marbles, and g green marbles. Each marble also has a unique label, so we can describe the probability of drawing a particular marble as well. Answer a variety of questions about the base probabilities, conditional probabilities, and AND probabilities (see a, b, and c above).

We refer to a processes that produces a random outcome, such as tossing The Unbiased Coin, drawing two cards from The Three-Card Deck, and selecting marbles from The Urn, as a **process**.[1]

Defining Chance

> The **chance** of something gives the percentage of times it is expected to happen, when the basic process is done over and over agin, independently and under the same conditions.

Suppose we repeat a process many times. The chance of soemthing is the proportion of times the event occurs. We can think of it this way: $P(\text{event}) = 100\%\frac{\text{number of events}}{\text{total number of experiments}}$ if the total number of experiments is very large (i.e., infinity).

Note that it is common to talk about chances using probabilities from 0 to 1 rather than percents from 0% to 100%. It happens that chance (in %) are a little easier to talk about in English (i.e., you have a 50% chance of winning), but probabilities are a little easier to work with mathematically. We'll tend to use chance (in %) because we're not go very deep into the math, and the percent is a bit more intuitive and easier to talk about. However, be aware that I might occassionally talk about probabilities. But it is easier to go back and forth between the two.

If all outcomes of a process are equally likely, such as Heads and Tails for The Unbiased Coin, then we can say that the chance of each outcome is $\frac{100\%}{\text{total number of outcomes}}$.[2] Finding the entire set of equally likely outcomes is sometimes a bit tricky, but in most cases, it's trivial.

We have two facts about chance:

1. Chances are between 0% and 100%.
2. The chance of something equals 100% minus the chance of the opposite thing.

[1] Some authors refer to these random processes as "experiments," but we avoid that language to avoid confusion with randomized experiments uses to draw causal inferences.

[2] Also equally likely: 3♣, the 9♢, and the A♢ in The Three Card Deck; drawing any particular marble (supposing for example, that each marble was labeled with a unique number); 1, 2, 3, 4, 5, and 6 when rolling a six-sided die.

Conditional Probabilites

Conditional probabilities put conditions into the problem. They ask for chance that something happens *given that something else is true*. For example, we might ask the following:

1. What is the chance of getting Head on our second toss of The Coin, *given that the first toss was a Head*?
2. What is the chance that the second card drawn from The Three-Card Deck is a ♣, *given that the first card drawn was a diamond*?
3. What is the probability that the 3rd marble drawn from The Urn is red, *given that the first two drawn were also red*?

Multiplication Rule

Sometimes, we are interested in AND chances. That is, we are interested in the chance that something will happen AND something else will happen. For example, the first card drawn from The Three-Card Deck is a ♣ AND the second card drawn from The Three-Card Deck is ♣.

Multiplication Rule: The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given that the first has happened.

In the technical notation, $P(\text{event 1 and event 2}) = P(\text{event 1})(\text{event 2} \mid \text{event 2})$. Note that we read the vertical bar | as "given."

Independence

Two things are **independent** if the chances for the second given the first are the same, no matter how the first one turns out. Otherwise, the two things are **dependent**.

When drawing at random **with replacement**, the draws are independent. Without replacement, the draws are dependent.

If two things are independent, then the chance that both will happen equal the product of their unconditional probabilities. This is a special case of the multiplicaiton rule.